UMO 2024 Questions

Q1. *(20 points)* What is the last non-zero digit of 15!?

(source: Enda 5)

Solution: We can compute the last non-zero digit of 15! by writing out 15!, cancelling every possible factor of 10 and then computing the last digit by doing multiplication modulo 10.

$$\begin{aligned} 15! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 3 \cdot 5 \\ &= 10^3 \cdot 3 \cdot 4 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 11 \cdot 12 \cdot 13 \cdot 7 \cdot 3 \\ &= 10^3 \cdot (10n + 8) \end{aligned}$$

for some nonnegative integer n, by computing the product of the terms not divisible by 10 on the second line modulo 10.

Q2. *(20 points)* Dissect a square into three triangles. What's the smallest possible difference between the areas of the largest and smallest triangles?
(Give your answer in the form $\frac{a}{b}$, where a, b, c, d are positive integers such that $\gcd(a, b) = 1$.*)*

(source: Chris 5)

Solution: Intuitively, the dissection must be of the form $\{\triangle ABX, \triangle XCD, \triangle AXD\}$ where X is a point on side BC of square $ABCD$. (See below for a proper explanation of why this is true). Then $\triangle AXD$ has area $\frac{1}{2}$ and the other two triangles have an average area of $\frac{1}{4}$, meaning that the smaller one has area difference with $\triangle AXD$ of at least $\frac{1}{4}$.

Proper explanation: the dissection gives us a planar graph with $F = 4$, so $E - V = 2$. E is the number of edges of the triangles, not double counting shared edges; the triangulation is necessarily connected and so at least $3 - 1 = 2$ edges are shared. Thus $E \leq 7$ and so $V \leq 5$, but $V = 4$ clearly doesn't work and so $V = 5$. The fifth vertex cannot be in the interior of the square or else a triangulation is clearly not possible. Therefore it lies on a side; without loss of generality it lies on BC.

Q3. *(20 points)* Let X be a random variable with $S_X \subseteq [-1, 1]$. If ν is the largest possible value of $\mathrm{Var}(X)$, submit 720ν.

(source: Blaise 9)

Solution: From the definition of variance

$$\mathrm{Var}(X) = \mathbf{E}(X^2) - \mathbf{E}(X)^2$$

Since $-1 \leq X \leq 1$ then $0 \leq X^2 \leq 1$ so $= \mathbf{E}(X^2) \leq 1$. As squares are non-negative $\mathbf{E}(X)^2 \geq 0$. Hence $\mathrm{Var}(X) \leq 1$. But obviously if X has pmf $p_X(x) = \begin{cases} \frac{1}{2} & x \in \{-1, 1\} \\ 0 & \text{otherwise} \end{cases}$ then $\mathrm{Var}(X) = 1$ and so $\nu = 1$. Hence $720\nu = 720$.

Remark. it is also well known (e.g. this is a continuous variant of the lemma in post #2 on the AoPS thread for USAMO 2020/6) that the range of a random variable is at least twice its standard deviation.

Q4. *(20 points)* Suppose the following system of equations holds:

🍉 + 🍍 = 🍒
🍒 + 🍑 = 🍌
🍍 + 🍒 = 🍆
🍑 + 🍆 = 🍉
🍆 + 🍉 = 🍒
🍉 + 🍒 = 🍍

What is the sum of all possible values of 🍒 + 🍌 + 🍆?

(source: Jamie 4)

Solution: This solution assumes that the fruits may be taken as variables representing real numbers, and that the addition operation is commutative.

Equations 1 and 6 imply that 🍍 = 🍒 $:= x$ and 🍉 $= 0$. Equation 5 then indicates that 🍆 $= x$; equation 4 indicates that 🍑 $= -x$ and equation 3 indicates that $2x = x \implies x = 0$. Then equation 2 implies that 🍌 $= 0$ as well, so all six variables equal zero. Hence the answer is 0.

Q5. *(25 points)* We can evaluate 5^2 by moving the 2 in front of the 5. Suppose this also works for the matrix $\begin{bmatrix} i & l \\ t & g \end{bmatrix}$. Find the sum of all possible values of $i + g$.

(source: Blaise 3)

Solution: We have that

$$\begin{bmatrix} 2i & 2l \\ 2t & 2g \end{bmatrix} = \begin{bmatrix} i & l \\ t & g \end{bmatrix}^2 = \begin{bmatrix} i^2+lt & il+lg \\ it+tg & lt+g^2 \end{bmatrix}.$$

The equations $2l = l(i+g)$, $2t = t(i+g)$ indicate that either $l = t = 0$ or $i + g = 2$. It is possible for $i+g=2$, such as if $i=2$, $l=t=g=0$. If $l=t=0$, we have $2i=i^2$, $2g=g^2$. Thus we can choose i, $g \in \{0, 2\}$ as we please (and it is easy to check these work as well), so we get $i+g \in \{0, 2, 4\}$. Thus the answer is $0+2+4=6$.

Q6. *(25 points)* Alfred Young has some used linear algebra tutorial sheets to throw away. The paper bin is only wide enough to fit two tutorial sheets side by side, but sheets can be stacked on top of each other in two piles.
Alfred assigns the number 3 to the bottom page in the taller pile, the one above it a 4 and so on. He then does the same with the smaller pile starting from 2.
Alfred then changes each page's value by dividing it by n, where n is the number of pages at or above its level in its own pile, incremented by one if and only if the page is in the larger pile and there is a page at its level in the smaller pile.
What is the product of all the page values if Alfred made two piles of 20, and then added 24 more pages to one of the piles (before doing any labelling or computation)?

(source: Jamie 1)

Solution: The initial labels give a product of $\frac{1}{2}46! \cdot 21!$. The division part then gives us a quotient of $20! \cdot 24! \cdot 26 \cdot 27 \cdot \cdots \cdot 45 = 20! \cdot \frac{45!}{25}$. Thus the answer ends up being $\frac{1}{2}21 \cdot 25 \cdot 46 = 12075$.

Q7. *(25 points)* When given a polygon, Tammy tries to paint its edges without painting the same vertex twice. However, Tammy is blind and she may accidently paint an edge adjacent to the one she intends to paint without realizing.
Given a 5D hypercube, how many edges can Tammy paint while guaranteeing that she doesn't paint the same edge twice?

(source: Jamie 5)

Solution: If Tammy selects an edge for painting, there are 8 other edges she could accidentally paint, blocking out 9 edges in total. As there are 80 edges on a 5D hypercube [see Question 8 for how to compute this], at most 8 edges can be selected safely. However, if you imagine painting the edges $(a, b, c, d, 0) \sim (a, b, c, d, 1)$ for the 8 tuples (a, b, c, d) with sum either 1 or 3, then the distances between the vertices of two distinct edges is at least 2, so no two selected edges are adjacent to a common edge. Thus the answer is 8.

Q8. *(25 points)* How many edges does a 7D hypercube have?

(source: Chris 11)

Solution: An n dimensional hypercube has 2^n vertices. n edges emanate from each vertex; each edge is counted exactly twice by this process. Thus an n dimensional hypercube has $n2^{n-1}$ edges, which is 448 for $n = 7$.

Q9. *(30 points)* Compute

$$\int_0^\pi \cos(t)\sin(\cos(t)) - (\sin(t) + \sin^2(t))\cos(\cos(t))\, \mathrm{d}t.$$

Give your answer in the form $a \sin b$, where a, b are integers with b nonnegative.

(source: Chris 3)

Solution: By recognition,

$$\int \cos(t)\sin(\cos(t)) - (\sin(t) + \sin^2(t))\cos(\cos(t))\, \mathrm{d}t = (1 + \sin(t))\sin(\cos(t)) + C,\ C \in \mathbb{R}$$

and so the answer is $(1 + \sin\pi)\sin(\cos\pi) - (1 + \sin 0)\sin(\cos 0) = \sin(-1) - \sin(1) = -2\sin(1)$.

Q10. *(30 points)* Find an integer $0 \le d \le 36$ such that $2^{28} - d$ is divisible by 37.

(source: Enda 3)

Solution: We can write it out: $2, 4, 8, 16, 32 \equiv -5, -10, -20, -40 \equiv -3, -6, -12, -24, -48 \equiv -11, -22, -44 \equiv -7, -14, -28, -56 \equiv -19, -38 \equiv -1$ tells us that $2^{18} \equiv -1 \pmod{37}$ and so $2^{28} \equiv -2^{10} \equiv 12 \pmod{37}$ so that $d = 12$ works.

Remark. One can more easily obtain that $2^{18} \equiv -1 \pmod{37}$ by noting that 2 is a quadratic residue for primes that are $\pm 1 \pmod 8$, which 37 is not.

Q11. *(30 points)* A *cyclone* is a sequence of subsets of a finite set, where each is a proper subset of the one after it. A *complete* cyclone is one where the ith subset has exactly i elements, for each subset in the cyclone. Suppose you have a complete cyclone for a set S of size 24,

$$\{s_1\}, \{s_1, s_2\}, \ldots, \{s_1, \ldots, s_{23}\}, S.$$

You can change any one set in the cyclone at a time, while maintaining the property that it is a cyclone. Submit the minimum number of changes to reverse the cyclone into

$$\{s_{24}\}, \{s_{24}, s_{23}\}, \ldots, \{s_{24}, \ldots, s_2\}, S.$$

(source: Blaise 6)

Solution: Observe that introducing extra elements does not help, because we can simply delete annything involving these during the operations and this does not increase the number of total changes. Because we are using (sub)sets which involve distinct elements, the final element is always S, and one element is removed in each step backwards; represent a cyclone by this sequence instead. It's clear that if the subsets are $T_1, \ldots, T_{24} := S$ then the only valid operation to T_i is to remove the element of it not in T_{i-1}, and add the element of T_{i+1} which was not previously in it. This corresponds to swapping two elements in the sequence representation. The original cyclone and the final cyclone are $\binom{24}{2} = 276$ inversions apart, so this is the answer. **Q12.** *(30 points)* Find the sum of all integers

$n \geq 4$ with the following property:

> There exist $n-1$ integers which can be written in a circle, such that the set of products of adjacent numbers is an $n-1$ element subset of $\{1, 2, \ldots, n\}$.

(source: David 1)

Solution: By strong Bertrand, there are two primes between n and $\frac{36}{25}n$ for $n \geq 25$, meaning that for $n \geq 36$ there are at least two primes between $\frac{n}{2}$ and n. We can also manually check this holds for all $n \geq 4$, except for $n = 4,\ 6,\ 10$. Note that our subset of products cannot contain a prime $p > \frac{n}{2}$, since p can only be represented as $1 \cdot p$, and then the other neighbour of p in the circle cannot produce p again (since the $n-1$ element subset has all distinct elements) but all other multiples of p exceed n. Thus if there are two such primes, it must be impossible. Thus it is not possible unless $n = 4,\ 6,\ 10$ but by inspection $n = 4$ is clearly impossible. Valid constructions for $n = 6$ and $n = 10$ are $(2, 2, 1, 1, 3)$ and $(2, 3, 3, 1, 5, 2, 4, 1, 1)$ respectively.

Q13. *(35 points)* Five horses take part in a race. How many ways can the horses finish if arbitrary ties are allowed?

(source: David 3)

Solution: We separate into cases based on partitions of 5 as representations of tied groupings.

- $5 = 1 + 1 + 1 + 1 + 1$ represents no ties, which contributes $5! = 120$ finishes.
- $5 = 1 + 1 + 1 + 2$ represents one two-horse tie, which contributes $\binom{5}{2} \cdot 4 \cdot 3! = 240$ finishes (select the two horses to tie, select what ranking they receive, permute the remaining horses)
- $5 = 1 + 1 + 3$ represents one three-horse tie, which contributes $\binom{5}{3} \cdot 3 \cdot 2! = 60$ finishes (select the three horses to tie, select what ranking they receive, permute the remaining horses)

- $5 = 1 + 4$ represents one four-horse tie, which contributes $\binom{5}{4} \cdot 2 = 10$ finishes (select the four horses to tie, select if they come first or second)
- $5 = 5$ represents one five-horse tie, which contributes 1 finish.
- $5 = 1+2+2$ represents two two-horse ties, which contributes $\binom{5}{2} \cdot \binom{3}{2} \cdot 3 = 90$ finishes (select the higher-ranked pair, then the lower ranked pair, then the arrangement of pairs and individual)
- $5 = 2 + 3$ represents one two-horse tie and one three-horse tie, which contributes $\binom{5}{2} \cdot 2 = 20$ finishes (select the tied pair, then whether they come first or second)

The total number of possibilities is $120 + 240 + 60 + 10 + 1 + 90 + 20 = 541$ finishes.

Remark. It's well known that the answer for n horses is $\sum_{m=1}^{\infty} \frac{m^n}{2^{m+1}}$. (Sequence A000670 in the OEIS)

Q14. *(35 points)* For positive integers n and k, define $M_{n,k}$ to be the $k \times k$ matrix such that for all $1 \le i, j \le k$, the (i, j)th entry is $n^{\frac{i+j}{2}-1}$. Find the sum of all positive integers n for which there exists a positive integer k such that $M_{n,k}^{31} = 31M_{n,k}^{30}$.

(source: Chris 1)

Solution: Note that $M_{n,k}^2 = (1 + n + \cdots + n^{k-1})M_{n,k}$ and so we must have $31 = 1 + n + \cdots + n^{k-1}$. Note that $k \ge 2$ since $M_{n,1} = \begin{bmatrix}1\end{bmatrix}$ the 1×1 identity matrix which clearly doesn't work, and $(n, k) = (1, 31)$ works. Otherwise, $n \ge 2 \implies 31 > n^{k-1} \ge 2^{k-1}$ and so $k \le 5$. We now manually check, and find solutions $(2, 5), (5, 3), (30, 2)$ for an answer of $1 + 2 + 5 + 30 = 38$.

Q15. *(35 points)* When given a polygon, Timmy tries to paint its vertices without painting the same vertex twice. However, Timmy is blind and he may accidently paint a vertex adjacent to the one he intends to paint without realizing.
Given a 7D hypercube, how many vertices can Timmy paint while guaranteeing that he doesn't paint the same vertex twice?

(source: Dougal 1)

Solution: When Timmy aims to paint a vertex, there are 7 other vertices he could paint. Thus one attempt blocks out 8 possible vertices. Thus he can paint at most $\frac{2^7}{8} = 16$ times. It turns out that this Hamming code is perfect and so the answer is 16.

Q16. *(35 points)* A cubic polynomial P has each of its coefficients uniformly and independently sampled from $\{1, 2, \ldots, 100\}$. What is the probability that there does not exist an integer n such that $P(n)$ is divisible by 5?

Give your answer in the form $\frac{a}{b}$, where a, b are positive integers with $\gcd(a, b) = 1$.

(source: David 2)

Solution: Note that we might as well be sampling from residues modulo 5. It suffices to do inclusion-exclusion computations to find the number of polynomials P having any residue modulo 5 as a root. Noting that only one cubic polynomial P (considered modulo 5) has all five residues modulo 5 as a root (the zero polynomial), but otherwise in order to guarantee d given roots we are simply forced to make 1 out of 5^d selections, due to 5 being prime meaning that the system solving works out, we thus get

$$\binom{5}{1}\frac{1}{5} - \binom{5}{2}\frac{1}{25} + \binom{5}{3}\frac{1}{125} - \binom{5}{4}\frac{1}{625} + \frac{1}{625} = \frac{421}{625}$$

as the probability that P *does* have a root modulo 5. Thus the answer is its complement, $\frac{204}{625}$.

Q17. *(40 points)*

$$\begin{aligned}
&x \in X, S : X \to X \\
&\forall m, n \in \mathbb{N} \cup \{0\}, m \neq n \implies S^m(x) \neq S^n(x) \\
&N = \{S^n(x) | n \in \mathbb{N} \cup \{0\}\} \\
&P : N \times N \to N, \forall m, n \in \mathbb{N} \cup \{0\}, P(S^m(x), S^n(x)) := S^{m+n}(x) \\
&\forall m \in \mathbb{N} \cup \{0\}, P_m : N \to N, P_m(y) := P(S^m(x), y) \\
&M : N \times N \to N, \forall m, n \in \mathbb{N} \cup \{0\}, M(S^m(x), S^n(x)) := P_m^n(x) \\
&\forall y \in N, L(y) = \{z \in N | (\exists w \in N, w \neq x)(P(z, w) = y)\} \\
&I = \{y \in N | (\nexists w, z \in L(y))(M(w, z) = y))\} \\
&H = \{S^n(x) | n \in \mathbb{N} \cup \{0\}, n \leq 100\} \\
&|I \cap H| =?
\end{aligned}$$

(source: Chris 7)

Solution: Treat S as a successor function. N reconstructs $\mathbb{Z}_{\geq 0}$, P reconstructs addition of two numbers, P_m reconstructs the function $x \mapsto x + m$, M reconstructs multiplication, $L(y)$ is the set of numbers less than y, I represents the set of primes and H is the set of numbers at most 100. Thus $|I \cap H|$ is the number of primes at most 100, which is 25.

Q18. *(40 points)* How many multisets of positive integers A are there with sum of elements 2022 such that for each $1 \leq m \leq 2022$ there is a unique sub-multiset B of A such that the sum of the elements of B is m?

(source: David 4)

Solution: First, note that A has exactly 2023 sub-multisets (because exactly one sub-multiset has sum 0, and every other sub-multiset has sum a positive integer at most 2022 and so is counted in the 2022 unique multisets with sums 1, 2, ..., 2022). Thus if its elements $a_1 < \cdots < a_m$ appear $k_1, \ldots, k_m$ times, then $\sum a_i k_i = 2022$ and $\prod(k_i + 1) = 2023$.

Fix a selection $(k_1, \ldots, k_m)$. We claim that for all $1 \le j \le m-1$, $\sum_{i=1}^{j} a_i k_i = a_{i+1} - 1$, by induction. Observe that 1 must be an element of A, and so it is a_1. Note that all of the a_i, $i \ge 2$, must be larger than k_1 or else some sum of 1*s* will equal a_i, contradiction to uniqueness. However, if none of them equal $k_1 + 1$, then that sum is unachievable. Thus $a_2 = k_1 + 1$, the base case of $j = 1$. Assume that the inductive statement holds for $j = \ell$. Observe that $(a_1, \ldots, a_\ell)$ and $(k_1, \ldots, k_\ell)$ allow one to construct any number from 1 to $\sum_{i=1}^{\ell} a_i k_i := S$. Then for all $i > \ell$, $a_i > S$ or else we have a contradiction to uniqueness of sums. But if no $a_i = S + 1$, then $S + 1$ will be unachievable. Thus $a_{\ell+1} = S + 1$, proving the statement for $j = \ell + 1$ and completing the induction.

Since we know a_1, this means that the k_i actually uniquely determine the a_i, and so there is exactly one valid A for each selection of the k_i. Thus we only need to determine the number of ways to factorise 2023 with order. These are 2023, 7×289, 289×7, 17×119, 119×17, $7 \times 17 \times 17$, $17 \times 7 \times 17$, $7 \times 17 \times 17$ for a total of 8 such A. **Q19.** *(40 points)*

In the following, all polygons have their vertices listed in anticlockwise order.
Sharky is playing with his favourite square $ABCD$. Today, he decides to draw a triangle ADE, and two squares $EDFG$ and $AEHI$. To his surprise, the sum of the areas of the three polygons he drew is equal to the area of his favourite square.
Let α be the (positive) angle measure of $\angle AED$ in radians. Compute $\lfloor 50\alpha \rfloor$.

(source: Chris 9)

Solution: Let a, b, c be the lengths of sides AD, DE, EA. Then $\frac{1}{2}bc\sin(\alpha) + b^2 + c^2 = a^2 = b^2 + c^2 - 2bc\cos(\alpha)$ and so $\sin(\alpha) = -4\cos(\alpha)$. Thus $\cos(\alpha) = -\frac{1}{\sqrt{17}}$, so $\sin\left(\alpha - \frac{\pi}{2}\right) = \frac{1}{\sqrt{17}} \approx 0.2425$. Using $\sin(x) \approx x - \frac{x^3}{6}$, α is a bit less than $\frac{\pi}{2} + 0.245 \approx 1.816$ (In fact, it's about 1.8158). Thus $\lfloor 50\alpha \rfloor = 90$.

Remark. It's only needed to use $\sin(x) \le x$ and $\frac{1}{\sqrt{17}} < 0.25$ with both of these approximations relatively close to get that $1.8 < \alpha < 1.82$ is reasonable.

Q20. *(40 points)* You have seven coins in a line, of which five show heads and two show tails. Each minute, you select a random number $x \in \{1, 2, 3, 4, 5, 6, 7\}$ (uniformly distributed) and flip the xth coin from the left. What is the expected number of flips until all seven coins simultaneously show heads (for the first time)?

(source: Chris 8)

Solution: Let n replace 7 and let E_i be the answer when 5 is replaced by i. Also denote $x_i = E_i - E_{i+1}$. Then $x_0 = 1$ because if zero coins show heads, any flip will gain a head. Note $E_i = \frac{i}{n}(E_{i-1}+1) + \frac{n-i}{n}(E_{i+1}+1)$ so $\frac{n-i}{n}(E_i - E_{i+1}) = \frac{i}{n}(E_{i-1} - E_i) + 1$. Hence $x_i = \frac{i}{n-i}x_{i-1} + \frac{n}{n-i}$. It follows that

$$x_{n-1} = \sum_{i=0}^{n-1} \frac{n}{n-i} \prod_{j=i+1}^{n-1} \frac{j}{n-j} = \sum_{i=0}^{n-1} \frac{n!}{i!(n-i)!} = 2^n - 1$$

and $x_{n-2} = \frac{1}{n-1}(x_{n-1} - n) = \frac{2^n-n-1}{n-1}$. Letting $n = 7$, since $E_7 = 0$, $E_5 = x_5 + x_6 = 147$.

Q21. *(40 points)*
Compute

$$\frac{1}{\sqrt{13}}\left(\left(\frac{-3+\sqrt{13}}{2}\right)^6 - \left(\frac{-3-\sqrt{13}}{2}\right)^6\right).$$

(source: Enda 4)

Solution: Let $(x_i)_{i=0}^{\infty}$ be a sequence defined by $x_0 = 0$, $x_1 = 1$, and for $n \geq 2$, $x_n = x_{n-2} - 3x_{n-1}$. Then the characteristic equation for this linear recurrence is $\lambda^2 + 3\lambda - 1 = 0$, so $\lambda = \frac{-3\pm\sqrt{13}}{2}$. It's well known that therefore $x_n = A\left(\frac{-3+\sqrt{13}}{2}\right)^n + B\left(\frac{-3-\sqrt{13}}{2}\right)^n$ for some constants A, B. $x_0 = 0$ implies that $A + B = 0$, while $x_1 = 1$ implies that $-\frac{3}{2}(A+B) + \frac{\sqrt{13}}{2}(A-B) = 1$, so that $A - B = \frac{2}{\sqrt{13}}$ and so $A = \frac{1}{\sqrt{13}}$, $B = -\frac{1}{\sqrt{13}}$. We are asked to compute x_6. We have $x_2 = -3, x_3 = 10, x_4 = -33, x_5 = 109, x_6 = -360$ as required.

Q22. *(Up to 42 points)* Submit a positive integer k, as well as an arrangement of m primes in a circle such that for any two primes p, q next to each other in the circle, $pq = x^2 + x + k$ for some positive integer x. You will be scored based on when you submit and the value of m (earlier and larger are better, respectively).

(source: Chris 6)

Solution: IMO 2022/3 asks students to prove that for a set S of odd prime numbers, and fixed positive integer k, there is at most one way to arrange the elements of S in a circle with the property described in the question. It appears that this statement is quite silly, since surely large rings of primes like this do not exist? Well, see post #30 at `https://artofproblemsolving.com/community/c6h2883213p25635143` for a set S with $k = 41$ and $m = 385$. (Funnily enough, $n^2 + n + 41$ the "prime-generating polynomial" makes an appearance!)