Q1. *(20 points)* What is the last non-zero digit of 15!?

(source: Enda 5)

Attempt 1: __________ Attempt 2: __________ Attempt 3: __________

Q2. *(20 points)* Dissect a square into three triangles. What's the smallest possible difference between the areas of the largest and smallest triangles?

(Give your answer in the form $\frac{a}{b}$, where a, b, c, d are positive integers such that $\gcd(a, b) = 1$*.)*

(source: Chris 5)

Attempt 1: __________ Attempt 2: __________ Attempt 3: __________

Q3. *(20 points)* Let X be a random variable with $S_X \subseteq [-1, 1]$. If ν is the largest possible value of $\mathrm{Var}(X)$, submit 720ν.

(source: Blaise 9)

Attempt 1: ____________ Attempt 2: ____________ Attempt 3: ____________

Q4. *(20 points)* Suppose the following system of equations holds:

🍉 + 🍍 = 🍒

🍒 + 🍑 = 🍌

🍍 + 🍒 = 🍆

🍑 + 🍆 = 🍉

🍆 + 🍉 = 🍒

🍉 + 🍒 = 🍍

What is the sum of all possible values of 🍒 + 🍌 + 🍆?

(source: Jamie 4)

Attempt 1: ____________ Attempt 2: ____________ Attempt 3: ____________

Q5. *(25 points)* We can evaluate 5^2 by moving the 2 in front of the 5. Suppose this also works for the matrix $\begin{bmatrix} i & l \\ t & g \end{bmatrix}$. Find the sum of all possible values of $i + g$.

(source: Blaise 3)

Attempt 1: __________ Attempt 2: __________ Attempt 3: __________

Q6. *(25 points)* Alfred Young has some used linear algebra tutorial sheets to throw away. The paper bin is only wide enough to fit two tutorial sheets side by side, but sheets can be stacked on top of each other in two piles.
Alfred assigns the number 3 to the bottom page in the taller pile, the one above it a 4 and so on. He then does the same with the smaller pile starting from 2.
Alfred then changes each page's value by dividing it by n, where n is the number of pages at or above its level in its own pile, incremented by one if and only if the page is in the larger pile and there is a page at its level in the smaller pile.
What is the product of all the page values if Alfred made two piles of 20, and then added 24 more pages to one of the piles (before doing any labelling or computation)?

(source: Jamie 1)

Attempt 1: __________ Attempt 2: __________ Attempt 3: __________

Q7. *(25 points)* When given a polygon, Tammy tries to paint its edges without painting the same vertex twice. However, Tammy is blind and she may accidently paint an edge adjacent to the one she intends to paint without realizing.
Given a 5D hypercube, how many edges can Tammy paint while guaranteeing that she doesn't paint the same edge twice?

(source: Jamie 5)

Attempt 1: __________ Attempt 2: __________ Attempt 3: __________

Q8. *(25 points)* How many edges does a 7D hypercube have?

(source: Chris 11)

Attempt 1: __________ Attempt 2: __________ Attempt 3: __________

Q9. *(30 points)* Compute

$$\int_0^{\pi} \cos(t)\sin(\cos(t)) - (\sin(t) + \sin^2(t))\cos(\cos(t))\ \mathrm{d}t.$$

Give your answer in the form $a \sin b$, where a, b are integers with b nonnegative.

(source: Chris 3)

Attempt 1: ________________ Attempt 2: ________________ Attempt 3: ________________

Q10. *(30 points)* Find an integer $0 \leq d \leq 36$ such that $2^{28} - d$ is divisible by 37.

(source: Enda 3)

Attempt 1: ________________ Attempt 2: ________________ Attempt 3: ________________

Q11. *(30 points)* A *cyclone* is a sequence of subsets of a finite set, where each is a proper subset of the one after it. A *complete* cyclone is one where the ith subset has exactly i elements, for each subset in the cyclone. Suppose you have a complete cyclone for a set S of size 24,

$$\{s_1\}, \{s_1, s_2\}, \ldots, \{s_1, \ldots, s_{23}\}, S.$$

You can change any one set in the cyclone at a time, while maintaining the property that it is a cyclone. Submit the minimum number of changes to reverse the cyclone into

$$\{s_{24}\}, \{s_{24}, s_{23}\}, \ldots, \{s_{24}, \ldots, s_2\}, S.$$

(source: Blaise 6)

Attempt 1: ______________ Attempt 2: ______________ Attempt 3: ______________

Q12. *(30 points)* Find the sum of all integers $n \geq 4$ with the following property:

> There exist $n-1$ integers which can be written in a circle, such that the set of products of adjacent numbers is an $n-1$ element subset of $\{1, 2, \ldots, n\}$.

(source: David 1)

Attempt 1: ______________ Attempt 2: ______________ Attempt 3: ______________

Q13. *(35 points)* Five horses take part in a race. How many ways can the horses finish if arbitrary ties are allowed?

(source: David 3)

Attempt 1: __________ Attempt 2: __________ Attempt 3: __________

Q14. *(35 points)* For positive integers n and k, define $M_{n,k}$ to be the $k \times k$ matrix such that for all $1 \leq i, j \leq k$, the (i, j)th entry is $n^{\frac{i+j}{2}-1}$. Find the sum of all positive integers n for which there exists a positive integer k such that $M_{n,k}^{31} = 31M_{n,k}^{30}$.

(source: Chris 1)

Attempt 1: __________ Attempt 2: __________ Attempt 3: __________

Q15. *(35 points)* When given a polygon, Timmy tries to paint its vertices without painting the same vertex twice. However, Timmy is blind and he may accidently paint a vertex adjacent to the one he intends to paint without realizing.
Given a 7D hypercube, how many vertices can Timmy paint while guaranteeing that he doesn't paint the same vertex twice?

(source: Dougal 1)

Attempt 1: __________ Attempt 2: __________ Attempt 3: __________

Q16. *(35 points)* A cubic polynomial P has each of its coefficients uniformly and independently sampled from $\{1, 2, \ldots, 100\}$. What is the probability that there does not exist an integer n such that $P(n)$ is divisible by 5?
Give your answer in the form $\frac{a}{b}$, *where* a, b *are positive integers with* $\gcd(a, b) = 1$.

(source: David 2)

Attempt 1: __________ Attempt 2: __________ Attempt 3: __________

Q17. *(40 points)*

$$
\begin{aligned}
&x \in X, S : X \to X \\
&\forall m, n \in \mathbb{N} \cup \{0\}, m \neq n \implies S^m(x) \neq S^n(x) \\
&N = \{S^n(x) | n \in \mathbb{N} \cup \{0\}\} \\
&P : N \times N \to N, \forall m, n \in \mathbb{N} \cup \{0\}, P(S^m(x), S^n(x)) := S^{m+n}(x) \\
&\forall m \in \mathbb{N} \cup \{0\}, P_m : N \to N, P_m(y) := P(S^m(x), y) \\
&M : N \times N \to N, \forall m, n \in \mathbb{N} \cup \{0\}, M(S^m(x), S^n(x)) := P_m^n(x) \\
&\forall y \in N, L(y) = \{z \in N | (\exists w \in N, w \neq x)(P(z, w) = y)\} \\
&I = \{y \in N | (\nexists w, z \in L(y))(M(w, z) = y))\} \\
&H = \{S^n(x) | n \in \mathbb{N} \cup \{0\}, n \leq 100\} \\
&|I \cap H| =?
\end{aligned}
$$

(source: Chris 7)

Attempt 1: ____________ Attempt 2: ____________ Attempt 3: ____________

Q18. *(40 points)* How many multisets of positive integers A are there with sum of elements 2022 such that for each $1 \leq m \leq 2022$ there is a unique sub-multiset B of A such that the sum of the elements of B is m?

(source: David 4)

Attempt 1: ____________ Attempt 2: ____________ Attempt 3: ____________

Q19. *(40 points)* In the following, all polygons have their vertices listed in anticlockwise order. Sharky is playing with his favourite square $ABCD$. Today, he decides to draw a triangle ADE, and two squares $EDFG$ and $AEHI$. To his surprise, the sum of the areas of the three polygons he drew is equal to the area of his favourite square.
Let α be the (positive) angle measure of $\angle AED$ in radians. Compute $\lfloor 50\alpha \rfloor$.

(source: Chris 9)

Attempt 1: __________ Attempt 2: __________ Attempt 3: __________

Q20. *(40 points)* You have seven coins in a line, of which five show heads and two show tails. Each minute, you select a random number $x \in \{1, 2, 3, 4, 5, 6, 7\}$ (uniformly distributed) and flip the xth coin from the left. What is the expected number of flips until all seven coins simultaneously show heads (for the first time)?

(source: Chris 8)

Attempt 1: __________ Attempt 2: __________ Attempt 3: __________

Q21. *(40 points)*
Compute

$$\frac{1}{\sqrt{13}}\left(\left(\frac{-3+\sqrt{13}}{2}\right)^6-\left(\frac{-3-\sqrt{13}}{2}\right)^6\right).$$

(source: Enda 4)

Attempt 1: __________ Attempt 2: __________ Attempt 3: __________

Q22. *(Up to 42 points)* Submit a positive integer k, as well as an arrangement of m primes in a circle such that for any two primes p, q next to each other in the circle, $pq = x^2 + x + k$ for some positive integer x. You will be scored based on when you submit and the value of m (earlier and larger are better, respectively).

(source: Chris 6)

Attempt 1: __________ Attempt 2: __________ Attempt 3: __________