PARADOX

Welcome to the final issue of *Paradox* for the year. We do not have our regular *Paradox* competition this time, but do not despair! We have news of an exciting T–shirt competition to tickle the fancies of any mathematically inclined designers out there. Also there is a report on the recent Maths Olympics, solutions to the previous competition problems, some problems and jokes, and we take a look at how Archimedes treated infinite series.

Comments from you are still valuable—*Paradox* will be back next year and your say will have an influence on future issues. They can either be placed in the *Paradox* box (opposite the Maths office) or e-mailed to: `paradox@maths.mu.oz.au`. Thanks again to contributors and proof-readers —in particular Lawrence Ip, Vanessa Teague and Tony Wirth. Good luck to all doing exams or theses and have a terrific stress-free summer.

Chaitanya Rao, *Paradox* Editor.

T-SHIRT COMPETITION!

MUMS is looking for a design for a T-shirt. The design must be somewhat 'mathematical' in nature, contain the phrase "Melbourne University Mathematical Society" and must be something that people would actually want to buy and wear—nothing vulgar, please!!! All designs will be judged according to originality, imagination and creativity, colour (no limitations) and mathematical content (it must contain some). The winning designer will be awarded **$50**. All entries must be in by the middle of February, 1997. So all of you with a creative bent may want to use your energy productively and win some money along the way. Entries may be placed in the *Paradox* box opposite the Maths office or mailed to:

Melbourne University Mathematical Society
Department of Mathematics
The University of Melbourne
Parkville, VIC 3052

HIGH SCHOOL WINS MATHS OLYMPICS

For the second time in five years (indeed in the history of the competition) a secondary school team captured first prize. "The New Dark Blue" from Melbourne Grammar School pipped "Beyond Unlimited" on countback. The last time a school team showed such lack of respect for their elders was in 1992.

At the beginning of the competition, a runner from each team dashed athletically down to the front of Theatre A and collected a question, then ran back up the stairs to solve it with other members of their team. The physical and intellectual challenge of sprinting up and down the aisles without colliding into people was at least as great as the challenge of actually solving the question, especially for those strange students who hadn't spent hours beforehand practising running up and down the stairs.

The students who entered proved that maths is not always as civilised and sedate as it may appear. A member of the Happy Farmers sustained a black eye (we're sure it was an accident), and one of Charlie's Angels collided with your favourite *Paradox* editor.

Professor White acted as the M.C., proposing mathematical problems for the many spectators to solve and then throwing Mars bars into the crowd when the questions were successfully answered. Thus the audience was entertained with a mixture of mathematics and projectile confectionary.

"Bananas" showed up, "Pyjamas" didn't. They were substituted by members of an enthusiastic audience. "No Real Solutions" entered for the sixth, and possibly last time, placing in the top half for the first time. There were a record 41 entries this year. With only 28 places, this left many teams disappointed. Get in early next year!

Scores of the top ten teams are listed below. In the event of equal scores, the number of the most recently solved question was used to separate teams.

(1)	The New Dark Blue	95	(=6)	Charlie's Angels	30
(2)	Beyond Unlimited	95	(=6)	Division by Zero	30
(3)	We Don't Count	75	(8)	The Incompetents	30
(4)	Pandits	45	(9)	Happy Farmers and One Disgruntled Capitalist	30
(5)	Muffin and Friends	40	(10)	Beyond Infinity	25

Many thanks to Warwick Evers, who set the questions and Dugal Ure, who did most of the organisation. To see the full set of results, as well as the questions and answers, you can go to the newly formed MUMS homepage at `http://www.maths.mu.oz.au/~lip/mums/mums.html`

PROBLEMS AND SOLUTIONS TO PREVIOUS PARADOX COMPETITION

From the last competition, problems 2 and 4 proved to be the most popular with quite a few correct solutions received. Unfortunately no winners could be found for the other two problems, but the solutions to all four problems are included here. Solutions by the winners are not reproduced word for word, but reflect the approach used.

I know that people were particularly intrigued by the first problem, so here is that solution first:

Problem 1: Given 13 distinct real numbers, show there exist two, say x and y, satisfying:

$$\frac{xy+1}{x-y} > \frac{7}{2}$$

Solution: (no winners)

We can write the thirteen numbers as $\tan\theta_1, \tan\theta_2, \ldots, \tan\theta_{13}$, where $-\frac{\pi}{2} < \theta_i < \frac{\pi}{2}, (i = 1, 2, \ldots, 13)$. Consider the numbers θ_1 through to θ_{13}. As these thirteen numbers all lie within an interval of length π there exist two of them, say θ_m and θ_n with

$$\begin{aligned} 0 < \theta_m - \theta_n &< \frac{\pi}{12} \\ \Rightarrow \tan(\theta_m - \theta_n) &< \tan\frac{\pi}{12} \text{ (as tan is an increasing function on our interval)} \\ &= \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ \text{ie. } \frac{\tan\theta_m - \tan\theta_n}{1 + \tan\theta_m \tan\theta_n} &< \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{4}} \\ &= 2 - \sqrt{3} \end{aligned}$$

Letting $x = \tan\theta_m$ and $y = \tan\theta_n$ (two of our original thirteen numbers) this equation becomes

$$\begin{aligned} \frac{x-y}{1+xy} &< 2-\sqrt{3} \\ \Rightarrow \frac{xy+1}{x-y} &> 2+\sqrt{3} \\ &> \frac{7}{2} \text{ as required.} \end{aligned}$$

Nice problem, isn't it? The key is in recognising that after reciprocating both sides of the original inequality, the left side resembles the form $\tan(a-b)$.

Problem 2: The numbers $0, 1, 0, 1, 0$ and 0 are written clockwise around the circumference of a circle. It is possible to make "moves" in each of which we add 1 to each number of a certain pair of adjacent numbers. Is it possible by means of finitely many such moves to make all the numbers on the circumference equal?

Solution: (Winner: Nick Nethercote)

Assume it is possible to make all the numbers equal after finitely many moves. Let u, v, w, x, y, z be the number of times we add 1 to each of the pairs labelled a and b, b and c, $\dots$, f and a respectively (see Figure 1). Then the numbers at each of these positions after the moves are made are $0+z+u$, $1+u+v$, $0+v+w$, $1+w+x$, $0+x+y$ and $0+y+z$.

These must all be equal for some values of u through to z. Hence

$$0+z+u = 1+u+v \Rightarrow z = v+1$$

$$0+v+w = 1+w+x \Rightarrow v = x+1$$

$$0+x+y = 0+y+z \Rightarrow x = z$$

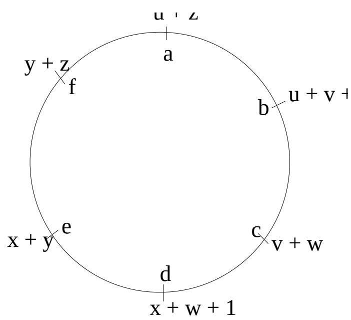

Figure 1: Numbers on the circle after the moves are made

This gives $x = z = v+1 = x+2$, a clear contradiction. We conclude that it is not possible to make all the numbers equal.

Problem 3: A student is shocked to find that he only has 37 days to prepare for the upcoming Maths Olympics. From past experience he knows that he will require no more than 60 hours of "training" (problem-solving, running up and down stairs, who knows?). He also wishes to train at least 1 hour per day. Show that no matter how he organises his schedule, there is a succession of days during which he trains exactly 13 hours (here we are assuming that he is training for a whole number of hours per day).

Solution: (no winners)

Suppose the student trains a_i hours up to and including the ith day. Then since he trains at least hour per day, $a_1 < a_2 < \cdots < a_{37}$. We are also told that $a_{37} \leq 60$, so $a_{37} + 13 \leq 73$.

Consider the 74 numbers $a_1, a_2, \ldots, a_{37}, a_1 + 13, a_2 + 13, \ldots, a_{37} + 13$. These are whole numbers between 1 and 73 inclusive—hence there must be at least two of them which are equal. Therefore $a_k = a_j + 13$ for some $k > j$, so he has trained precisely 13 hours from day $j + 1$ to day k (inclusive).

Problem 4: Evaluate the sum

$$\sum_{n=1}^{1996} \frac{n^2 + n + 1}{n!}$$

Note from the Editor: Unfortunately there was a typographical error in the above sum. The sum was intended to be $\sum_{n=1}^{1996} (-1)^n \frac{n^2+n+1}{n!}$. Apologies to all, especially to the student who produced a three page-long fractional answer! The best one can do with the problem as it stands is approximate the solution by evaluation of the infinite series (as several people did).

Solution: (Winner: Trung Tran)

$$\begin{aligned} \sum_{n=1}^{1996} \frac{n^2 + n + 1}{n!} &= \sum_{n=1}^{1996} \frac{n+1}{(n-1)!} + \sum_{n=1}^{1996} \frac{1}{n!} \\ &= \sum_{n=1}^{1996} \frac{n-1}{(n-1)!} + \sum_{n=1}^{1996} \frac{2}{(n-1)!} + \sum_{n=1}^{1996} \frac{1}{n!} \end{aligned}$$

Letting $n - 1 = m$ in the first two sums and approximating $\sum_{n=0}^{N} \frac{1}{n!}$ by e for $N = 1995$ or 1996 (the approximation is good to hundreds of decimal places) we find

$$\begin{aligned} \sum_{n=1}^{1996} \frac{n^2 + n + 1}{n!} &= \sum_{m=0}^{1995} \frac{m}{m!} + \sum_{m=0}^{1995} \frac{2}{m!} + \sum_{n=1}^{1996} \frac{1}{n!} \\ &= \sum_{m=1}^{1995} \frac{1}{(m-1)!} + 2\sum_{m=0}^{1995} \frac{1}{m!} + \sum_{n=0}^{1996} \frac{1}{n!} - \frac{1}{0!} \\ &\approx e + 2e + e - 1 \\ &= 4e - 1 \end{aligned}$$

SOME PROBLEMS

In case of withdrawal symptoms from a lack of maths during the summer, have a go at these problems!

1. Evaluate the sum

$$\sum_{n=1}^{1996}(-1)^n\frac{n^2+n+1}{n!}$$

2. Find all integer solutions to: $6x^2+2y^2=z^2$.

3. A rectangle is cut into five rectangular pieces of equal area. Prove that at least two of the pieces are identical.

4. If f is a function $f:\mathbf{R}^+\to\mathbf{R}^+$ ($\mathbf{R}^+$ is the positive reals) with the property that $\forall x\in\mathbf{R}^+, f(f(x))=1/x$, then show that f cannot be continuous.

THE PRISONER'S PARADOX

The following paradox was first posed in the 1940s and since then has generated a vast literature. As with many paradoxes, there is no consensus on a single "right" answer.

A prisoner has been sentenced on Saturday. The judge announces that "the hanging" will take place at noon on one of the seven days of next week, but he will not know which day it is until he is told on the morning of the day of the hanging. The prisoner, on mulling this over, decided that the judge's sentence could not possibly be carried out.

"For example," said he, "I can't be hung next Saturday, the last day of the week, because on Friday afternoon I'd still be alive and I'd know for sure that I'd be hung on Saturday. But I'd know this *before* I was told about it—Saturday morning—and this would contradict the judge's statement." In the same way, he argued, they could not hang him on Friday, or Thursday, Wednesday, Tuesday or Monday. "And they can't hang me tomorrow," thought the prisoner, "because I know it today!"

What (if anything) is wrong with this reasoning?

A COUPLE OF TRUE STORIES

The following problem can be solved either the easy way or the hard way.

Two trains 200 kilometres apart are moving toward each other; each one is going at a speed of 50 kilometres per hour. A fly starting on the front of one of them flies back and forth between them at a rate of 75 kilometres per hour. It does this until the trains collide and crush the fly to death. What is the total distance the fly has flown?

The fly actually hits each train an infinite number of times before it gets crushed, and one could solve the problem the hard way with pencil and paper by summing an infinite series of distances. The easy way is as follows: since the trains are 200 kilometres apart and each train is going 50 kilometres an hour, it takes 2 hours for the trains to collide. Therefore the fly was flying for two hours. Since the fly was flying at a rate of 75 kilometres per hour, the fly must have flown 150 kilometres. That's all there is to it.

When this problem was posed to John von Neumann (a mathematician best known for having developed game theory), he immediately replied, "150 kilometres."

"It is very strange," said the poser, "but nearly everyone tries to sum the infinite series."

“What do you mean, strange?” asked von Neumann. “That’s how I did it!”

Enrico Fermi (a famous theoretical physicist), while studying in college, was bored by his math classes. He walked up to the professor and said, “My classes are too easy!” The professor looked at him, and said, “Well, I’m sure you’ll find this interesting.” Then the professor copied nine problems from a book to a paper and gave the paper to Fermi. A month later, the professor ran into Fermi, “So how are you doing with the problems I gave you?” “Oh, they are very hard. I only managed to solve six of them.” The professor was visibly shocked, “What!? But those are unsolved problems!”

SOME JOKES

—Why did the chicken cross the Moebius strip?
—To get to the other ... er, um ...

—What is a compact city?
—It’s a city that can be guarded by finitely many near-sighted policemen!

—What do you get if you cross an elephant with a mountain climber.
—You can’t do that. A mountain climber is a scalar.

There are two groups of people in the world: those who can be categorized into one of two groups of people, and those who can’t.

Engineers think that their equations are approximations to reality.
Physicists think that reality is an approximation to their equations.
Mathematicians don’t care.

SOME POETRY!

This limerick was written by Jon Saxton (an author of maths textbooks).

$$(12 + 144 + 20 + 3\sqrt{4})/7 + (5 \times 11) = 9^2 + 0$$

In other words:

A Dozen, a Gross and a Score,
Plus three times the square root of four,
Divided by seven,
Plus five times eleven,
Equals nine squared and not a bit more!

ARCHIMEDES AND THE GEOMETRIC SERIES—by Tony Wirth

Nowadays we take it for granted that we can say:

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2,$$

by applying the formula:

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}.$$

Of course substituting $x = -1$, which gives

$$1 - 1 + 1 - 1 + \cdots = \frac{1}{1-(-1)} = \frac{1}{2}$$

makes no sense, but it was not until a couple of hundred years ago that people really understood why. It is instructive to examine the approach that Archimedes of Syracuse took, to this issue. With little contemporary understanding of sequences and series, he proceeded to determine that value of a geometric series with great care.

Archimedes of Syracuse (287-212 BC), son of an astronomer, studied at Alexandria in his youth. Following his education he returned to his home town where he stayed for the rest of his life. He was a well known figure in the Hellenic world, perhaps for such feats as creating a parabolic mirror which focussed the sun's rays on to the approaching Roman fleet and burnt their ships. Interestingly, because he was so famous for his inventions, his mathematical abilities, considered on a par with Newton and Gauss, were often overshadowed.

Almost four hundred pages of his work—often consisting of small articles—have survived. Some are Greek manuscripts, others Latin translations such as those by Tartaglia of 1543. Of particular relevance to this discussion is his work in calculating the area of a segment of a parabola. Archimedes had two approaches to solving this problem. The first, detailed in a work called *The Method*, was only found in a library in Constantinople in 1906. It applies the principles of mechanics to derive an expression for the area of the segment. Another book *Quadrature of the Parabola* contains a more geometrical and rigorous method. It is here that we find his approach to geometric series.

Archimedes shows that one can approximate the area of a parabolic segment to within any given accuracy by inscribing a sequence of triangles of maximal area (see Figure 2). A little work shows that the area of each term in the sequence (which consists of 2^n triangles) is one quarter the previous term. Furthermore, no matter how many triangles are constructed the sum of their areas will never equal that of the parabolic segment. So the series of the areas is:

$$|\triangle PRT| \left(1 + \frac{1}{4} + \frac{1}{16} + \cdots\right).$$

But Archimedes only ever considered a finite number of such terms. He first showed (indirectly) that:

$$1 + \frac{1}{4} + \frac{1}{16} + \cdots + \left(\frac{1}{4}\right)^n + \frac{1}{3}\left(\frac{1}{4}\right)^n = \frac{4}{3} \qquad (\star)$$

Now, he claims that the area of the parabolic segment (which we call A) is four-thirds the area of the largest inscribed triangle with the same base (call this K).

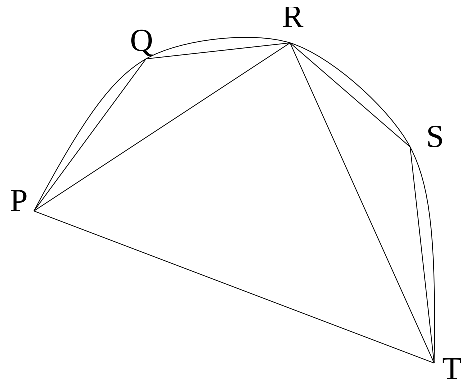

Figure 2: Inscribing maximal triangles in a parabolic segment

Proof by contradiction: if not, then the segment area is either greater than or less than four-thirds the triangle area.

Case 1: $A > K$.

Then it is possible to construct sufficiently many triangles, so that S the sum of these lies between K and A. This means that $S > K$. But ($\star$) says that $S < K$ no matter how many terms we add to the series. Contradiction.

Case 2: $A < K$.

The area of each term of the sequence of the triangles A_n diminishes "geometrically" to zero. Hence there will be some term A_i for which $A_i < K - A$. But we know from ($\star$) that

$$A_1 + A_2 + \cdots + A_i + \frac{A_i}{3} = K.$$

But then,

$$K - (A_1 + A_2 + \cdots + A_i) = \frac{A_i}{3} < A_i < K - A.$$

Hence

$$A < A_1 + A_2 + \cdots + A_i$$

which is clearly nonsense as the triangles lie *within* the parabolic segment and the sum of their areas cannot exceed that of the segment. Contradiction.

So it must be that $A = K$ and Archimedes' claim is true.

Notice how tedious and careful the argument is. Surely no one would wish to reproduce this kind of proof for every geometric series and clealy no-one does these days. However we are fortunate enough to live after considerable work has been done in the understanding of sequences and series, and so such issues are conveniently ignored.

BIBLIOGRAPHY

Heath, T. L. (ed.), *The works of Archimedes with The Method of Archimedes*, Dover, New York, 1953.

Kline, M., *Mathematical thought from Ancient to Modern times*, Oxford University Press, New York, 1972.