Revision Workshop
Calculus 2 (MAST10006)

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If L, M, a, $k \in \mathbb{R}$ and f, g are functions $\mathbb{R} \to \mathbb{R}$ with

$$\lim_{x\to a} f(x) = L, \quad \lim_{x\to a} g(x) = M$$

(the existence of these limits is crucial) then we have

$$\lim_{x\to a}(f(x) \pm g(x)) = L \pm M$$
$$\lim_{x\to a} f(x) \cdot g(x)^{\pm 1} = L \cdot M^{\pm 1}$$
$$\lim_{x\to a} k \cdot f(x) = kL$$

Continuity: $h\colon \mathbb{R} \to \mathbb{R}$ is continuous at a if

$$\lim_{x\to a} h(x) = f(a)$$

(always write this - it's the definition). Most of our favourite functions are continuous over their domain, like polynomials, e^x, log, trigonometric functions.

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Continuity: if f, g are functions $\mathbb{R} \to \mathbb{R}$ with f continuous at $a \in \mathbb{R}$, then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

(passing the limit into the function).
L'Hopital's rule: if f, g are differentiable (except perhaps at some $a \in \mathbb{R}$), and $\frac{f(x)}{g(x)}$ is indeterminate at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Sandwich theorem: If f, g, h are continuous at $x = a$ and $f(x) \leq g(x) \leq h(x)$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \implies \lim_{x \to a} g(x) = L.$$

(Tip: most commonly used when $L = 0$).

(a)

$$\lim_{\theta\to 0} \cos\left(\frac{\cosh(\theta)-1}{\theta}\right)$$

(b)

$$\lim_{x\to 0} x^2 \tanh\left(\frac{1}{x}\right)$$

(c) Determine the continuity of $f(x)$ at $x = 0$, where

$$f(x) = \begin{cases} \cos\left(\frac{\cosh(\theta)-1}{\theta}\right) & x < 0 \\ a & x = 0 \\ x^2 \tanh\left(\frac{1}{x}\right) & x > 0 \end{cases}$$

(a) Bring the limit into the cos and then use L'Hopital. (Answer: 1)

$$\begin{aligned}
&\lim_{\theta\to 0}\cos\left(\frac{\cosh(\theta)-1}{\theta}\right) \\
&=\cos\left(\lim_{\theta\to 0}\frac{\cosh(\theta)-1}{\theta}\right) && \text{by continuity of cos} \\
&=\cos\left(\lim_{\theta\to 0}\frac{\sinh(\theta)}{1}\right) && \text{by l'Hopital's rule } \left(\frac{0}{0}\right) \\
&=\cos(\sinh(0)) && \text{by limit laws and continuity of sinh} \\
&=\cos(0)=1
\end{aligned}$$

Justifications are essential for marks! E.g. on assignment 1, this was +2 marks for the result and +1 mark each for stating l'Hopital's rule, limit laws, and continuity of cosine/sinh.

(b) Sandwich theorem: remember $-1 \leq \tanh(\text{anything}) \leq 1$. (Answer: 0)

(c) Left- and right-hand limits differ, so the limit does not exist at the point. **By the definition of continuity** (state it), not continuous.

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Most basic lower-bounding technique: every term is at least some nonzero thing.

Question 2: Find all $c \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} \arctan(cn)$ converges.

Most basic upper-bounding technique: upper bound terms by some series you know converges.

Question 3: Determine the convergence of the following series:

(a)

$$\sum_{n=1}^{\infty} \frac{3n^2 + \cos^2(n) + 2n}{4n^5 + n^2 - 1}$$

(b)

$$\sum_{n=1}^{\infty} n^{-n} n!$$

Question 2

- If $c > 0$, $\arctan(cn) \geq \arctan(c) > 0$ for $n \geq 1$ so sum goes to ∞; similarly, if $c < 0$, sum goes to $-\infty$
- If $c = 0$ it's a sum of zeroes...

Question 3

(a) Convergent

- Notice that basically numerator degree $<$ denominator degree
- Numerator $\leq 6n^2$ and denominator $\geq 4n^5$ so sum is $\frac{3}{2}\times$ of sum of $\frac{1}{n^3}$ which converges (p-series with $p > 1$!)

(b) Example marks provided based on mid-semester.
Since $a_n = \frac{n!}{n^n} > 0$ for all $n \geq 1$, we can use ratio test. (+2 marks for checking these terms are greater than 0, +2 marks for stating we can use ratio test.)
We have

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)!}{(n+1)^{(n+1)}} \frac{(n)^{(n)}}{(n)!} \\ &= \frac{(n+1)n!}{n!} \frac{n^n}{(n+1)(n+1)^n} \\ &= \frac{n^n}{(n+1)^n}\end{aligned}$$

(+3 marks for correct ratio)

So

$$
\begin{aligned}
L &= \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \\
&= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \\
&= \lim_{n\to\infty} \left(\frac{1}{(1+\frac{1}{n})^n}\right) \\
&= \frac{1}{\lim_{n\to\infty}(1+\frac{1}{n})^n} && \text{by limit law} \\
&= \frac{1}{e} < 1 && \text{standard limit}
\end{aligned}
$$

(+3 for the correct limit result, +2 for stating that L is less than 1.)
Therefore by ratio test, the series $\sum_{i=1}^{\infty} \frac{n!}{n^n}$ converges. (+1 for re-stating ratio test, +3 for the correct answer (convergence).)

1. Definitions
   - $\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right), \cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$
   - All others defined analogously to circular trig functions
2. Key properties
   - $\cosh(x) \geq 1$, $-1 \leq \tanh(x) \geq 1$ since $|\cosh(x)| \geq |\sinh(x)|$ (follows from Pythagorean identity)
3. Using the formula sheet
   - Formula sheet has definitions of sinh, cosh and logarithmic formulas for arcsinh, arccosh, arctanh (VERY USEFUL)

- Question 4(b): Prove $\text{arcsech}(x) = \text{arctanh}(\sqrt{1-x^2})$
- Question 5: Solve $\cosh(x) + \sinh(x) = -2022$ over $x \in \mathbb{R}$ and then $x \in \mathbb{C}$

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- Question 4(b): Prove $\operatorname{sech}\left(\operatorname{arctanh}(\sqrt{1-x^2})\right) = x$ using formula sheet and definitions (lots of algebra - **make sure you are careful about definitions**)
- Question 5: $\cosh(x) + \sinh(x) = e^x$ for all $x \in \mathbb{C}$
  (a) If $x \in \mathbb{R}$, $e^x > 0 > -2022$, no solutions
  (b) If $x \in \mathbb{C}$, write $x = a + bi$; $|e^{bi}| = 1$ so $|e^a| = 2022 \implies a = \log(2022)$.
  Looking at $e^{bi} = cos(b) + i\sin(b) = -1$ (formula sheet), $b = (2k+1)\pi$, $k \in \mathbb{Z}$.

$e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$ are the real and imaginary parts of $e^{(a+bi)x} = e^{ax}\operatorname{cis}(bx)$. Note that we can swap $\frac{\mathrm{d}}{\mathrm{d}x}$, $\int$ with Im, Re; thus we can bring integrals and derivatives inside; the integral and derivative of $e^{\alpha x}$ where $\alpha \in \mathbb{C}$ is the same as normal.

Integration by parts:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

should be viewed as allowing you to differentiate some terrible function u at the cost of integrating some (better) function $\frac{\mathrm{d}v}{\mathrm{d}x}$.

Common mnemonic LIATE for which function to differentiate first - **L**ogarithms, **I**nverse (trig), **A**lgebraic (polynomials), **T**rig, **E**xponentials

Substitution theorem: If g is a “nice” function (injective, differentiable), then we can substitute $t = g(x)$ to get

$$\int f(g(x))\frac{\mathrm{d}t}{\mathrm{d}x}\mathrm{d}x = \int f(t)\ \mathrm{d}t$$

where if we integrated from a to b we now integrate from $g(a)$ to $g(b)$. We can also do this when g is secretly some inverse function, like if we want to let $x = \sin(t)$ in an integral we can do $t = \arcsin(x)$.

- "Ad hoc substitutions"
  - Things like $u = P(x)$ (P some polynomial), $u = \log(x)$, $u = \tan(x)$, etc.
  - Do this when the integral seems to be "in terms of" this function
- Trig substitution recommendations (below table reproduced from MAST10019 notes)

Problem Term	**Substitution**	**Domain**
$(c^2 + x^2)^k,\ k \in \mathbb{Z}$	$x = c\tan(t)$	$\mathbb{R}$
$(c^2 + x^2)^k,\ k \notin \mathbb{Z}$	$x = c\sinh(t)$	$\mathbb{R}$
$(c^2 - x^2)^k,\ k \in \mathbb{Z}$	It's a polynomial...	$\mathbb{R}\backslash\{-c, c\}$
$(c^2 - x^2)^k,\ k \notin \mathbb{Z}$	$x = c\sin(t)$	$(-c, c)$
$(x^2 - c^2)^k,\ k \in \mathbb{Z}$	It's a polynomial...	$\mathbb{R}\backslash\{-c, c\}$
$(x^2 - c^2)^k,\ k \notin \mathbb{Z}$	$x = c\cosh(t)$	$(-\infty, c) \cup (c, \infty)$

(a) (Explicitly with the complex exponential)

$$\int e^{-2x} \sin(5x) \, \mathrm{d}x$$

(b)

$$\int \sqrt{9 + x^2} \, \mathrm{d}x$$

(c)

$$\int x^2 \log\left(x^2\right) \, \mathrm{d}x$$

(a) Imaginary part of $e^{(5i-2)x}$; integrate. Then expand $e^{(5i-2)x} = e^{-2x}(cos(5x) + i\sin(5x))$ and use complex conjugates for the $\frac{1}{5i-2}$ term. **Make sure initial constant of integration is complex** $(+c + di)$

(b) As per the table, let $x = 3\sinh(t)$; once you get $\frac{9}{4}\sinh(2t) + \frac{9t}{2} + C$, use double angle formula and Pythagorean identity to get

$$\sinh(2t) = 2\sinh(t)\sqrt{\sinh^2(t) + 1}$$

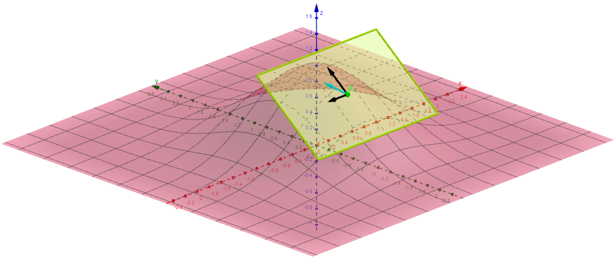
before substituting back $t = \operatorname{arcsinh}\left(\frac{x}{3}\right)$.

(c) Integrate by parts: differentiate $u = \log\left(x^2\right)$ and integrate $\frac{\mathrm{d}v}{\mathrm{d}x} = x^2$ as per LIATE

Multivariable functions

Directional derivative, tangent planes

- Partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ tell us the slope in the x- and y-directions: $\nabla f = (f_x, f_y)$.
- Tangent plane is uniquely determined by slopes in these directions: tangent plane at $(x, y) = (a, b)$ is $z = f_x(a, b)(x - a) + f_y(a, b)(y - b) + f(a, b)$.
- Directional derivative is really just direction on the plane. But this has a neat formula: $\mathbf{D}_{\hat{\mathbf{u}}} f = \nabla f \cdot \hat{\mathbf{u}}$.



- Finding critical points
  - These are the points where the tangent plane is horizontal, i.e. parallel to the xy-plane. $z = c$ for some c
  - Find these by solving $f_x = f_y = 0$.
- Classifying critical points
  - Hessian is $H_f(x, y) = f_{xx} f_{yy} - f_{xy}^2$ (remember $f_{xy} = f_{yx}$)
  - If $H_f < 0$, saddle point.
    If $H_f > 0$, local extremum: then check f_{xx}, if $f_{xx} > 0$ then local min and if $f_{xx} < 0$ then local max; the second derivative test!
    Note that since $H_f > 0$, $f_{xx} f_{yy} > f_{xy}^2 \geq 0$, thus f_{xx}, f_{yy} have the same sign so you could do this for y if you wanted.
    If $H_f = 0$, test inconclusive.

- Question 14:
  (a) Find the directional derivative in direction towards origin at $(1, 5)$ if the tangent plane at $(1, 5)$ has equation $-2x + 3y - z = 17$
  (b) Find $\nabla f|_{(2022,-7)}$ if the directional derivatives at $(2022, -7)$ in directions $\theta = \frac{\pi}{4}$, $\phi = -\frac{\pi}{4}$ are 0, -8 respectively
- Question 13: Find and classify all critical points of $f(x, y) = \frac{1}{2}y^2 + \frac{1}{3}x^3y - xy + 2$.

- Question 14:
  (a) $z = -2x + 3y - 17$ implies $\nabla f = (-2, 3)$ at $(1, 5)$. Take dot product with unit direction vector $\frac{1}{\sqrt{26}}(-1, -5)$.
  (b) The unit direction vectors are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$; take dot product with $\nabla f := (a, b)$ and solve for a, b
- Question 13:
  - $f_x = x^2y - y$ and $f_y = y + \frac{1}{3}x^3 - x$.
    Solving $f_x = 0$ gives $y = 1$ or $x = \pm 1$. Solving $f_y = 0$ under these gives $(x, y) \in \{(0, 0), (\pm\sqrt{3}, 0), \left(1, \frac{2}{3}\right), \left(-1, -\frac{2}{3}\right)$.
  - $f_{xx} = 2xy$, $f_{yy} = 1$ and $f_{xy} = f_{yx} = x^2 - 1$.
    Hessian test tells us the first three are saddle points ($H_f = -1$, -4, -4 respectively), and the last two are local extrema ($H_f = \frac{4}{3}$ both times). Checking f_{xx} we get $\frac{4}{3}$ both times again so both are local minima.

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1. Separable equations, i.e. $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$
   Solution is to integrate: $\int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x$.
2. Integrating factor for first order linear, i.e. $\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)$
   Idea: want to multiply by some $I(x)$ so that the LHS $I(x)y' + I(x)P(x)y$ is the derivative of $(I(x)y)$. This means $I'(x) = I(x)P(x)$, which is a separable equation: solution is $I(x) = \exp\left(\int P(x)\,\mathrm{d}x\right)$ (don't need $+C$ because any antiderivative of P works)
   Now $(I(x)y)' = I(x)Q(x)$, thus

$$y = \frac{1}{I(x)} \int I(x)Q(x)\,\mathrm{d}x$$

   (the $+C$ is very important this time around)

3. Second-order linear with constant coefficients, i.e. $ay'' + by' + cy = f$.
    - Easier case: $f = 0$. $e^{\alpha x}$ has nice derivatives, so why don't we try that? LHS evaluates to $(a\alpha^2 + b\alpha + c)e^{\alpha x}$ and we realise we want α to be a solution of the *characteristic equation* $a\lambda^2 + b\lambda + c = 0$.
      Note that if the characteristic equation has a double root α we also get the solution $y = xe^{\alpha x}$, and if the roots are complex (say $\alpha \pm \beta i$) then via complex exponential we can find two solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$ instead.
      Once we have two different solutions, e.g. $e^{\alpha x}$, $e^{\beta x}$ (first case) we just have $y = Ae^{\alpha x} + Be^{\beta x}$.

3. Second-order linear with constant coefficients, continued
   - Harder case: $f \neq 0$. *Principle of superposition* says that the solution is $y = y_H + y_P$ where y_H solves the **H**omogenous version, and y_P is a random solution to the **P**articular equation. But what to guess for f? Well, there are some basic ideas. **Make sure you explain what guess you substituted**
     - If f is a polynomial of degree n, substitute y to be a generic polynomial of degree n.
     - If f is sin or cos, try something like $A\cos(x) + B\sin(x)$
     - If f is some $e^{\gamma x}$, guess $y = Ce^{\gamma x}$.
       Warning: if γ is already a root then this won't work (you'll get zero). Try $y = Cxe^{\gamma x}$ if it's a single root and $y = Cx^2e^{\gamma x}$ if it's a double root.

   Note that if f consists of the sum of more than one of these, then again by principle of superposition, you can split this up into more particular solutions, say $y_{P_1} + y_{P_2}$.

- Springs
  - Recall from physics: spring force $F_s = -kx$ and air resistance $R = -\beta v$, as well as gravity $W = mg$ and overall $F = ma$ where x, v, a are displacement, velocity, acceleration
  - Draw a diagram to get the ODE: it will be second order linear homogenous if the reference point is the equilibrium point, because then the equation becomes $ma = mg - k(s+y) - \beta y' \implies my'' + \beta y' + ky = 0$ (s is extension at equilibrium, y the distance below equilibrium point).
  - **Make sure you make good reference to first principles, i.e. Newton's law**

Damping and long term behaviour

The key here is to look at the characteristic equation! Recalling our second-order linear homogenous stuff, if the characteristic equation has:

- Two real roots: solution is $Ae^{at} + Be^{bt}$, called "overdamped" (spring stabilises to a point "too quickly"
- One real root: solution is $Ae^{\lambda t} + Bte^{\lambda t}$, "critically damped" (stops perfectly)
- Two complex roots: solution is $y = Ae^{at}\cos(bt) + Be^{at}\sin(bt)$, "underdamped" (oscillates as it comes to a stop)
- Special case: if $a = 0$ (i.e. $\beta = 0$, no damping): $y = A\cos(bt) + B\sin(bt)$ (simple harmonic motion)

Fun fact: can add a driving force (equivalent to $f \neq 0$, e.g. $f = \cos(\Omega t)$). Resonant frequency Ω breaks the spring.

- Question 7: Substitute $u = e^y$ into

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\operatorname{arctanh}(x)}{x^2 e^y} - \frac{2}{x}$$

and hence solve the ODE.

- Question 8: A beehive of 100,000 bees has a virus outbreak (no recovery, and no bees die). If $I(t)$ is the number (in thousands) of bees infected at time t days, and the rate of increase of I with respect to t is proportional to the product of the numbers of infected and uninfected bees at time t, then:
  - Show that
  $$\frac{\mathrm{d}I}{\mathrm{d}t} = \beta I(100 - I)$$
  for some β.
  - Solve the ODE, first generally, and then subject to 100 initial infections and 1000 total after 3 days.
  - Comment on $I(t)$ as $t \to \infty$.

- Question 7: substitution gives

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\operatorname{arctanh}(x)}{x^2} - \frac{2u}{x}$$

and this is clearly integrating factor. **Rearrange correctly!** When solving for u, will need to find $\int \operatorname{arctanh}(x)\, \mathrm{d}x$. Using integration by parts stuff from before, set $\frac{\mathrm{d}v}{\mathrm{d}x} = 1$. Don't forget to get back to y after finding u.

- Question 8: separable; I integral involves (simple) partial fractions. **Handle constant cases separately when solving.** Should get

$$I(t) = \frac{100}{1 + Ae^{-100\beta t}}$$

which then gets $A = 999$, $\beta = \frac{1}{300}\log\left(\frac{111}{11}\right)$. As $t \to \infty$, the Ae^{stuff} term disappears, so $I(t) \to 100$ which means every bee is infected.

- Question 10: Solve

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 2x + 18e^{5x}$$

  if $y(0) = \frac{1}{2}$, $y'(0) = 6$.
- Question 11: 4kg mass suspended from spring with spring constant k N m^{-1}; air resistance with damping constant $\beta = 12$ N s m^{-1} and gravity with $g = 9.8$ m s^{-2} (no other forces).
  - Derive the equation of motion.
  - Which of $k \in \{4, 8, 16\}$ would result in oscillations?
  - If $k = 4$, give an example of an extra external force $f(t)$ which would result in long-term constant amplitude oscillations (or explain why one doesn't exist)

- Question 10:
  - First solve homogenous: $\lambda^2 - 4\lambda + 4$ has double root $\lambda = 2$ so homogenous solution $y_H Ae^{2x} + Bxe^{2x}$.
  - First particular: $f = 2x$. Let $y_{P_1} = Cx + D$ and solve; $C = D = \frac{1}{2}$.
  - Second particular: $f = 18e^{5x}$. Let $y_{P_2} = Fe^{5x}$ and solve; $F = 2$.
  - Put it all together: $y = Ae^{2x} + Bxe^{2x} + \frac{x}{2} + \frac{1}{2} + 2e^{5x}$.
  - Use conditions $y(0) = \frac{1}{2}$, $y'(0) = 6$ to solve for A, B.

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- Question 11:
  - Substitute in the numbers: $4\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + ky = 0$.
  - Want underdamped, so discriminant of characteristic $144 - 16k < 0 \implies k > 9$; thus $k = 16$ only
  - For $k = 4$, overdamped (long-term stabilises position); add driving force such as $\cos(t)$ which induces a long-term constant sinusoidal oscillation (as by principle of superposition, the added $y_P = A\cos(t) + B\sin(t)$ for some A, B)

- Question 9: Sketch the solutions to $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin(y)$, $x \geq 0$, $0 \leq y \leq 4\pi$ without solving it. (Sketch the slope field and then trace the solutions, including equilibrium solutions $y = 0$, $y = \pi$ $y = 2\pi$, $y = 3\pi$, $y = 4\pi$)
- Question 12: Consider $f(x, y) = 6 - \sqrt{x^2 + y^2}$ and the surface S given by $z = f(x, y)$.
  - Find the level curve $z = 2$ and sketch it (solve and sketch $f(x, y) = 2$)
  - Find the level curve which passes through $(x, y) = (3, 4)$. (solve $f(x, y) = f(3, 4)$)
  - Sketch the cross-section of S in the xz-plane (set $y = 0$)
  - Sketch S (it's a cone)
- Any questions?