Hilbert's Hotel

Yi Zhern

Hilbert's Hotel is not your ordinary hotel. It is an idea proposed by German mathematician David Hilbert in the 1920s. The difference between Hilbert's Hotel and other hotels is not the service or the facilities. Hilbert's Hotel is a hotel with infinite rooms, and more interestingly, the hotel is fully booked by infinite customers.

Here's a question: As a receptionist, if a new customer comes in and asks for a room, what would you do? Can the new customer check in or do they have to find a new hotel? Well, the answer is yes- the new customer can check into this hotel despite it being fully booked.

"All guests, please change to the next room down, with room number 1 greater than your current room," you announce.

In other words, the guest with room number n is moved to the room with number $n+1$. You have now freed up room 1 for your new customer.

We can generalise this method to the situation when x new customers come, in which case each guest simply moves from room n to room $n+x$. This is a shocking result! Hilbert's fully booked hotel can still accept customers and earn some extra money!

Now consider another scenario. After successfully arranging the rooms for all the new guests, a bus arrives. The bus is infinitely long and carries an infinite number of new customers. Now, is it possible for you to rearrange guests so that everyone has somewhere to stay?

Stop here if you want to think about it for yourself…

It turns out the answer is yes once again.

"All guests, please move to the room with room number double your current room number," you announce over the loudspeaker.

The guest in room 1 moves to room 2, the guest in room 2 moves to room 4, and so on. In general, the guest in room number n, will move to room number $2n$. Consequently, all current guests have moved to an even-numbered room, leaving the odd-numbered rooms empty for the new arrivals! Since there are an infinite number of odd numbers, you can assign room $2n-1$ to the n^{th} new customer from the bus. And so, every guest from the infinitely long bus can check in to your fully booked hotel. As a receptionist, you now have an infinite amount of work to do as there is infinite number of new customers! If only you got to share in the hotel profits...

The situation gets more complicated when your hotel becomes more famous. One day, an infinite line of infinitely long buses arrives at the gates, all wanting to check in. As the receptionist, you can't reject them; you know that the money-hungry Hilbert will fire you if you do. So what can you do?

Stop here if you want to think about it first.

It turns out that you have two methods to choose from.

The first way uses the properties of primes.

"All guests, supposing you are in room number n, please move to the room with number 2^n," you announce. Then, you allocate the customer in seat t on the first bus to room number 3^t. Then, you allocate the customer in seat t on the second bus to room number 5^t. Following this pattern, every bus is allocated a unique prime number, and every customer in that particular bus is allocated to a room number that is a power of that prime number. As we know there are an infinite number of prime numbers (exercise: prove this by contradiction), we can accommodate every customer in all the infinitely many buses. The only problem with this approach is that there will be many empty rooms, such as 6 and 10, since they are not powers of primes. You know that Hilbert will be angry- empty rooms means losing money! So you think of a different approach.

The second method can solve this problem.

You open your computer and start to make a spreadsheet. As seen below, the first row of your spreadsheet represents your hotel's room numbers. Each subsequent row represents the seat numbers of the buses. The columns reflect the position of each customer in the bus/hotel:

	1	2	3	4	···
Hotel	B0S1	B0S2	B0S3	B0S4	···
Bus 1	B1S1	B1S2	B1S3	B1S4	···
Bus 2	B2S1	B2S2	B2S3	B2S4	···
Bus 3	B3S1	B3S2	B3S3	B3S4	···
⋮	⋮	⋮	⋮	⋮	⋱

Now, you imagine tying a string to each cell on your spreadsheet, starting at B0S1, then B0S2, B1S1, B2S1, B1S2, B0S3, B0S4, B1S3 and so on (refer to the next table). Then you pull that string out into a straight line. You will find that everyone is on this line! As a receptionist, you can just allocate the rooms according to the sequence of customers on this straight line, thus allowing you to allocate a room to everyone already in the hotel plus everyone on all infinite buses, without any rooms left empty. However, you may be a bit tired from knocking on each individual guests' room to ask them to move. But it's worth it, if Hilbert will let you keep your job.

	1	2	3	4	···
Hotel	B0S1	B0S2	B0S3	B0S4	···
Bus 1	B1S1	B1S2	B1S3	B1S4	···
Bus 2	B2S1	B2S2	B2S3	B2S4	···
Bus 3	B3S1	B3S2	B3S3	B3S4	···
⋮	⋮	⋮	⋮	⋮	⋱

In all three situations above, you found that even though your hotel is fully booked, you can still shuffle around guests to fit more people in. So far, based on this thought experiment, you might think that infinity always has the 'same size'; regardless of if you add a number, add an infinity or even add an infinite number of infinites, the infinity is still the same.
However, you quickly realise that you might not always be able to accommodate all guests.
Suppose that some other day, another infinitely long bus with an infinite number of customers arrives, however these customers are ordered by name rather than seat number. Their names are an infinite combination of A's and B's. You might think you can assign them to rooms the same way you did before. Interestingly, you can't. Stop here if you want to think about it first.

Here's why. Imagine you have already assigned everyone a room. You then list the rooms and the names of the corresponding guests. For example, imagine that the first three customers have names that start with AAAAA..., ABABAB..., and ABBBAA.... You then construct a new name. Starting with the first customer, you take the first letter of their name and change it to the opposite letter; take the second letter of the second customer's name and change it to the opposite letter, and so on. You do so infinitely, so that you have created another customer's name. Here, the first three letters of the new name are BAA.... Amazingly, you find that this new name is not on the list; by construction, it differs from every other customer in the hotel by at least one character. And so, this customer has not been allocated to a room. And there you have it- you haven't managed to fit every customer into the hotel! You start worrying about what Hilbert might say when he finds out...

Infinity can come in different sizes. In this thought experiment, while the hotel is infinite, it is what we call countably infinite, while the fourth situation has an uncountably infinite number of customers. There are plenty more unexpected facts when it comes to the size of infinity. Speaking informally, there are as many rational numbers (simple fractions) as there are integers. However, there are far more real numbers than there are rational numbers.

References

[1] Jeff Dekofsky. The infinite hotel paradox. https://www.youtube.com/watch?v=Uj3_KqkI9Zo, 2014.

[2] Veritasium. How an infinite hotel ran out of rooms. https://www.youtube.com/watch?v=OxGsU8oIWjY, 2021.

We hope you have enjoyed this first Paradox article of 2024. If you would like to contribute an article, or help us in any way, please contact MUMS and we will connect you with our Paradox Editor.