Revision Workshop
Linear Algebra

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A system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Can be written in matrix form

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Stick A and $\mathbf{b}$ together, and you get the augmented matrix of the system

$$[A|\mathbf{b}] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

To find solutions to the given system, we can apply the following operations on the augmented matrix

operation	notation	constraints
scale a row	$R_i \leftarrow \lambda R_i$	$\lambda \neq 0$
add a multiple of a row to another	$R_i \leftarrow R_i + \lambda R_j$	$i \neq j$
swap two rows	$R_i \leftrightarrow R_j$	$i \neq j$

Gaussian Elimination find the REF of any $n \times m$ matrix A,

1. if the top left entry is 0, swap the first row with another row with a non-zero first entry. If no such row exists, repeat on the $n \times (m-1)$ submatrix formed by removing the first column.
2. normalize the top left entry to 1 $R_1 \leftarrow \frac{1}{a_{11}} R_1$
3. Subtract the first row from every row i to obtain a 0 in its leading entry ($R_i \leftarrow R_i - a_{i1} R_1$ for each $i = 2 \ldots n$)
4. repeat on the $(n-1) \times (m-1)$ submatrix formed by removing the first row and column.

Gaussian-Jordan Elimination extends Gaussian Elimination to give us the RREF of A

1. Let i be the index of the last leading 1. Subtract the row i from every row $j < i$ to obtain a 0 on the column with the leading 1. ($R_i \leftarrow R_j - a_{j1} R_i$ for each $j = 1 ... i - 1$)
2. ignore all the columns after and including the column with the last leading 1.
3. repeat step 1.

Determinants

The determinant of a matrix A (denoted $|A|$ or $\det(A)$) has a very special property

- $\det(A) \neq 0$ if and only if A is invertible

Properties of determinants

1. $|I| = 1$
2. $|A| = \prod_{i=1}^{n} a_{ii}$ if A is triangular
3. $|A| = |B| \times |D|$ where $A = \left[\begin{array}{c|c} B & * \\ \hline 0 & D \end{array}\right]$, B and D are square.

Algebraic properties of determinants

1. $|A| = \left|A^T\right|$
2. $|AB| = |A| \times |B|$
3. $\left|A^{-1}\right| = |A|^{-1}$
4. $|kA| = k^n|A|$

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Row Reduction

Row operation	Effect
$R_i \leftrightarrow R_j$	$\det(A) = -\det(A')$
$R_i \leftarrow R_i + \lambda R_j$	$\det(A) = \det(A')$
$R_i \leftarrow \frac{1}{\lambda} R_i$	$\det(A) = \lambda \det(A')$

Note that the above operations work for **columns** in the same way that they work for rows.

Cofactor expansion

1. Select a row
2. For every term in this row: Calculate the determinant of the $(n-1) \times (n-1)$ matrix formed by removing the row and column containing our given term. Multiply this smaller determinant by the term itself. Multiply by -1 if in an even column.
3. Sum

Note that selecting a column also works.

Adjacency Matrices

On Construction (of the matrix):

- If there is an edge connecting vertex i to vertex j entry $A_{ij} = 1$ (or however many edges there are connecting the two vertices if you have a fancy graph).

BIG IMPORTANT THEOREM:

- Let A be the adjacency matrix of a given graph. Then the entries i, j of A^n counts n-steps walks from vertex i to j.

NB: To prove this, consider inducting on n.

Transpose and Inverse of Products

1. $(AB)^{-1} = B^{-1}A^{-1}$
2. $(AB)^T = B^T A^T$

A vector space $(V, +)$ over field K (typically $\mathbb{R}$ or $\mathbb{C}$) is a set with operation $+ : V \times V \to V$ such that:

1. $u + v \in V$
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. $\exists\, \mathbf{0} \in V$ such that $u + \mathbf{0} = \mathbf{0} + u = u$
5. $\forall\, u \in V\, \exists -u$ such that $u + (-u) = \mathbf{0}$
6. $ku \in V$
7. $(k + l)u = ku + lu$
8. $k(lu) = (kl)u$
9. $1 \times u = u$

for $u, v, w \in V$ and $k, l \in K$

Subspace Theorem/Lemma

W is a subspace of V if the following hold:

1. Set is non-empty
2. $u + v \in W$ for all $v, w \in W$
3. $ku \in W$ for all $u \in W$, $k \in \mathbb{R}$ (usually)

- We say a set of vectors $\{v_1, v_2, \ldots, v_n\}$ is linear independent iff:

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = 0 \implies a_1 = a_2 = \cdots = a_n = 0$$

and dependent if not.

- Any set of linearly independent vectors must have cardinality (size) at most the dimension of our vector space.

- A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is a basis for V iff:
  1. The set is linearly independent
  2. The set spans V (Every vector $v \in V$ can be expressed as $a_1 v_1 + \cdots + a_n v_n = v$ for some $a_1, \ldots, a_n \in \mathbb{R}$)
  3. We say that the number of vectors in any basis of a vector space is the *dimension* of the vector space.
- Any set of n linearly independent vectors is a basis for V.
- Any set of n vectors which spans V is a basis for V.
- Any set of r linearly independent vectors can be extended by adding vectors to form a basis.

- **Basis for Row space** is found by row reducing
- **Basis for Column space** is found by taking leading 1's in RRE form, identifying respective columns, then taking respective vectors in original columns.
- **Solution space** is the set of all solutions to $Ax = 0$ (also known as the kernel or null-space). This is found by row reducing and represented by taking a basis of the solution space.

Note that matrices and transformations are equivalent. Injectivity/Surjectivity of transformations can be deduced from leading 1s in the RREF:

- Every column has a leading 1 $\implies$ injective
- Every row has a leading 1 $\implies$ surjective

Change of Basis

- $P_{S\to B}$ (sending vector from standard to basis B)
- $P_{B\to S} = (P_{S\to B})^{-1}$ (B to standard)
- $P_{B\to B'} = P_{B\to S}P_{S\to B'}$
- $P_{B\to B'}[T]_B P_{B'\to B} = [T]'_B$ for any linear transformation T ($[T]_A$ means T represented in basis A)

- Principally we are finding solutions to the following equation:

$$Ax = \lambda x$$

  for $\lambda \in \mathbb{R}$
- It turns out that this is equivalent to solving

$$\det(A - \lambda I) = 0$$

  where I is the $n \times n$ identity matrix.

Characteristic Polynomials is exactly $\det(A - \lambda I)$.

1. By using co-factor expansion the enumeration of the determinant yields a polynomial in λ which we call the characteristic polynomial.
2. Solving for when this is $= 0$ gives us the eigenvalues (λ) for our matrix A.
3. We then substitute these values back in and solve for the solution space of $(A - \lambda I)v = 0$ to find our eigenvectors.

Diagonalisation:
We call a square matrix diagonalisable if there exists P such that $A = PDP^{-1}$ where D is a diagonalisable matrix.

- It turns out that a square matrix is diagonalisable iff there are n linearly independent eigenvectors.
- If we have such a set, we have that the columns of P are these vectors and D has diagonal entries that correspond to respective eigenvalues (you can verify yourself that this diagonalisation works).
- A corollary of this is that if we have n distinct eigenvalues then the matrix is diagonalisable - the converse isn't necessarily true though.

An inner product over vector space $(V, +)$ is a function $\langle \, , \, \rangle : V \times V \to K$ such that:

1. $\langle u, v \rangle = \overline{\langle v, u \rangle}$ (Don't worry about the conjugate line too much, the inner products we deal with are over the reals so it just ends up being $\langle u, v \rangle = \langle v, u \rangle$)
2. $\langle ku + lv, w \rangle = k\langle u, w \rangle + l\langle v, w \rangle$
3. $\langle v, v \rangle \geq 0$ where equality holds iff $v = \mathbf{0}$

where $v, w \in V$ and $k, l \in K$. A vector space with such an operation is called an inner product space.

WARNING: In the exam make sure to check every inner product you write $\langle \, , \, \rangle$ and not the dot product (unless specifically given)

This process allows us to construct orthonormal basis given a set of n linearly independent vectors:

$$u_1 = \frac{v_1}{||v_1||}$$

$$u_2 = \frac{v_2 - < v_2, u_1 > u_1}{||v_2 - < v_2, u_1 > u_1||}$$

$$u_3 = \frac{v_3 - < v_3, u_1 > u_1 - < v_3, u_2 > u_2}{||v_3 - < v_3, u_1 > u_1 - < v_3, u_2 > u_2||}$$

and so forth...
The best way of understand how this works is just to try it yourself! Maybe try finding an orthonormal basis using the set $\{(1, 2, 3), (0, 1, 5), (1, 0, 4)\}$ which spans $\mathbb{R}^3$.

An $n \times n$ matrix A is orthogonal, if and only if

- $AA^T = I$
- or equivalently, the columns of A are all orthogonal

If an $n \times n$ matrix A is symmetric, then

- A has all real eigenvalues
- A has all orthonormal eigenvectors

- Cross Product
- Line and Planes
- Cartesian and Vector Equations
- Distances

$$v = (a \cdot \hat{b})\hat{b}$$

  1. Point to Line
  2. Point to Plane
  3. Line to Plane
  4. Plane to Plane
  5. Line to Line
- Line of intersection between two planes.